# $H^*$-ALGEBRAS AND QUANTIZATION OF PARA-HERMITIAN SPACES

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Abstract. In the present note we describe a family of $H^*$-algebra structures on the set $L^2(X)$ of square integrable functions on a rank-one para-Hermitian symmetric space $X$.

## Introduction

Let $X$ be a para-Hermitian symmetric space of rank one. It is well-known that $X$ is isomorphic (up to a covering) to the quotient space $SL(n,\mathbb{R})/GL(n-1,\mathbb{R})$, see [4] for more details. We shall thus assume throughout this note that $X = G/H$, where $G = SL(n,\mathbb{R})$ and $H = GL(n-1,\mathbb{R})$.

The space $X$ allows the definition of a covariant symbolic calculus that generalizes the so-called convolution-first calculus on $\mathbb{R}^2$, see ([2, 7, 8]) for instance. Such a calculus, or quantization map $\mathrm{Op}_\sigma$, from the set of functions on $X$, called symbols, onto the set of linear operators acting on the representation space of the maximal degenerate series $\pi_{-\frac{n}{2}+i\sigma}$ of the group $G$, induces a non-commutative algebra structure on the set of symbols, that we suppose to be square integrable. On the other hand, the taking of the adjoint of an operator in such a calculus defines an involution on symbols. It turns out that these two data give rise to a $H^*$-algebra structure on $L^2(X)$.

According to the general theory, ([1, 5, 6]), every $H^*$-algebra is the direct orthogonal sum of its closed minimal two-sided ideals which are simple $H^*$-algebras. The main result of this note is the explicit description of such a decomposition for the Hilbert algebra $L^2(X)$ and its commutative subalgebra of $SO(n,\mathbb{R})$-invariants.

## 1. Definitions and basic facts

### 1.1. $H^*$-algebras.

**Definition 1.1.** *A set $R$ is called a $H^*$-algebra (or Hilbert algebra) if*

(1) *$R$ is a Banach algebra with involution;*
(2) *$R$ is a Hilbert space;*
(3) *the norm on the algebra $R$ coincides with the norm on the Hilbert space $R$;*
(4) *For all $x, y, z \in R$ one has $(xy, z) = (y, x^*z)$;*
(5) *For all $x \in R$ one has $\|x^*\| = \|x\|$;*
(6) *$xx^* \neq 0$ for $x \neq 0$.*

An example of a Hilbert algebra is the set of Hilbert-Schmidt operators $HS(I)$ that one can identify with the set of all matrices $(a_{\alpha\beta})$, where $\alpha, \beta$ belong to a fixed set of indices $I$, satisfying the condition $\sum_I |a_{\alpha\beta}|^2 < \infty$.

2000 *Mathematics Subject Classification.* 22E46,43A85,46B25.
*Key words and phrases.* Quantization, para-Hermitian symmetric spaces, Hilbert algebras.

**Theorem 1.2.** [6], *p. 331. Every Hilbert algebra is the direct orthogonal sum of its closed minimal two-sided ideals, which are simple Hilbert algebras.*

*Every simple Hilbert algebra is isomorphic to some algebra $HS(I)$ of Hilbert-Schmidt operators.*

**Definition 1.3.** [5], *p. 101 An idempotent $e \in R$ is said to be irreducible if it cannot be expressed as a sum $e = e_1 + e_2$ with $e_1$, $e_2$ idempotents which annihilate each other: $e_1 e_2 = e_2 e_1 = 0$.*

**Lemma 1.4.** [5], *p. 102. A subset $I$ of a Hilbert algebra $R$ is a minimal left (right) ideal if and only if it is of the form $I = R \cdot e$ ($I = e \cdot R$), where $e$ is an irreducible self-adjoint idempotent. Moreover $e \cdot R \cdot e$ is isomorphic to the set of complex numbers and $R$ is spanned by its minimal left ideals.*

Observe that any minimal left ideal is closed, since it is of the form $R \cdot e$.

**Corollary 1.5.** *If $R$ is a commutative Hilbert algebra, then any minimal left (or right) ideal is one-dimensional.*

1.2. **An algebra structure on $L^2(X)$.** Let $G = SL(n, \mathbb{R})$, $H = GL(n-1, \mathbb{R})$, $K = SO(n)$ and $M = SO(n-1)$. We consider $H$ as a subgroup of $G$, consisting of the matrices of the form $\begin{pmatrix} (\det h)^{-1} & 0 \\ 0 & h \end{pmatrix}$ with $h \in GL(n-1, \mathbb{R})$.

Let $P^-$ be the parabolic subgroup of $G$ consisting of $1 \times (n-1)$ lower block matrices $P = \begin{pmatrix} a & 0 \\ c & A \end{pmatrix}$, $a \in \mathbb{R}^*$, $c \in \mathbb{R}^{n-1}$ and $A \in GL(n-1, \mathbb{R})$ such that $a \cdot \det A = 1$. Similarly, let $P^+$ be the group of upper block matrices $P = \begin{pmatrix} a & b \\ 0 & A \end{pmatrix}$ $a \in \mathbb{R}^*$, $b \in \mathbb{R}^{n-1}$ and $A \in GL(n-1, \mathbb{R})$ such that $a \cdot \det A = 1$.

The group $G$ acts on the sphere $S = \{s \in \mathbb{R}^n, \|s\|^2 = 1\}$ and acts transitively on the set $\widetilde{S} = S/\sim$, where $s \sim s'$ if and only if $s = \pm s'$, by $g.s = \dfrac{g(s)}{\|g(s)\|}$, where $g(s)$ denotes the linear action of $G$ on $\mathbb{R}^n$. Clearly the stabilizer of the equivalence class of the first basis vector $\widetilde{e_1}$ is the group $P^+$, thus $\widetilde{S} \simeq G/P^+$. If $ds$ is the usual normalized surface measure on $S$, then $d(g.s) = \|g(s)\|^{-n} ds$.

For $\mu \in \mathbb{C}$, define the character $\omega_\mu$ of $P^\pm$ by $\omega_\mu(P) = |a|^\mu$. Consider the induced representations $\pi^\pm_\mu = \mathrm{Ind}^G_{P^\pm} \omega_{\mp\mu}$.

Both $\pi^+_\mu$ and $\pi^-_\mu$ can be realized on $C^\infty(\widetilde{S})$, the space of even smooth functions $\phi$ on $S$. This action is given by

$$\pi^+_\mu(g)\phi(s) = \phi(g^{-1}.s) \cdot \|g^{-1}(s)\|^\mu.$$

Let $\theta$ be the Cartan involution of $G$ given by $\theta(g) = {}^t g^{-1}$. Then

$$\pi^-_\mu(g)\phi(s) = \phi(\theta(g^{-1}).s) \cdot \|\theta(g^{-1})(s)\|^\mu.$$

Let $(\ ,\ )$ denote the usual inner product on $L^2(S)$ : $(\phi, \psi) = \int_S \phi(s)\bar{\psi}(s) ds$. Then this sesqui-linear form is invariant with respect to the pairs of representations $(\pi^+_\mu, \pi^+_{-\mu-n})$ and $(\pi^-_\mu, \pi^-_{-\mu-n})$. Therefore the representations $\pi^\pm_\mu$ are unitary for $\mathrm{Re}\,\mu = -\frac{n}{2}$.

The group $G$ acts also on $\widetilde{S} \times \widetilde{S}$ by

$$g(u, v) = (g.u, \theta(g)v). \tag{1}$$

This action is not transitive: the orbit $(\widetilde{S} \times \widetilde{S})^o = G.(\widetilde{e_1}, \widetilde{e_1}) = \{(u, v) : \langle u, v\rangle \neq 0\}/\sim$ is dense (here $\langle\ ,\ \rangle$ denotes the canonical inner product on $\mathbb{R}^n$). Moreover $(\widetilde{S} \times \widetilde{S})^o \simeq X$.

The map $f \mapsto f(u,v)|\langle u,v\rangle|^{-\frac{n}{2}+i\sigma}$, with $\sigma \in \mathbb{R}$ is a unitary $G$-isomorphism between $L^2(X)$ and $\pi^+_{-\frac{n}{2}+i\sigma} \widehat{\oplus}_2 \pi^-_{-\frac{n}{2}+i\sigma}$ acting on $L^2(\widetilde{S}\times\widetilde{S})$. The latter space is provided with the usual inner product.

Define the operator $A_\mu$ on $C^\infty(\widetilde{S})$ by the formula :

$$A_\mu\phi(s) = \int_S |\langle s,t\rangle|^{-\mu-n}\phi(t)dt.$$

This integral converges absolutely for $\mathrm{Re}\,\mu < -1$ and can be analytically extended to the whole complex plane as a meromorphic function of $\mu$. It is easily checked that $A_\mu$ is an intertwining operator, that is, $A_\mu\pi^\pm_\mu(g) = \pi^\mp_{-\mu-n}(g)A_\mu$.

The operator $A_{-\mu-n}\circ A_\mu$ intertwines the representation $\pi^\pm_\mu$ with itself and is therefore a scalar $c(\mu)\mathrm{Id}$ depending only on $\mu$. It can be computed using $K$-types.

Let $e(\mu) = \int_S |\langle s,t\rangle|^{-\mu-n}dt$, then $c(\mu) = e(\mu)e(-\mu-n)$. But on the other hand side $e(\mu) = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}}\frac{\Gamma\left(\frac{-\mu-n+1}{2}\right)}{\Gamma\left(-\frac{\mu}{2}\right)}$. One also shows that $A^*_\mu = A_{\bar{\mu}}$. So that, for $\mu = -\frac{n}{2}+i\sigma$ we get (by abuse of notations):

$$c(\sigma) = \frac{\Gamma\left(\frac{n}{2}\right)^2}{\pi}\cdot\frac{\Gamma\left(\frac{-n/2-i\sigma+1}{2}\right)\Gamma\left(\frac{-n/2+i\sigma+1}{2}\right)}{\Gamma\left(\frac{n/2+i\sigma}{2}\right)\Gamma\left(\frac{-n/2-i\sigma}{2}\right)},$$

and moreover $A_{-\frac{n}{2}+i\sigma}\circ A^*_{-\frac{n}{2}+i\sigma} = c(\sigma)\mathrm{Id}$, so that the operator $d(\sigma)A_{-\frac{n}{2}+i\sigma}$, where $d(\sigma) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)}\frac{\Gamma\left(\frac{n/2+i\sigma}{2}\right)}{\Gamma\left(\frac{-n/2+i\sigma+1}{2}\right)}$ is a unitary intertwiner between $\pi^-_{-\frac{n}{2}+i\sigma}$ and $\pi^+_{-\frac{n}{2}-i\sigma}$.

We thus get a $\pi^+_{-\frac{n}{2}+i\sigma}\widehat{\oplus}_2\,\bar{\pi}^+_{-\frac{n}{2}+i\sigma}$ invariant map from $L^2(X)$ onto $L^2(\widetilde{S}\times\widetilde{S})$ given by

$$f \mapsto d(\sigma)\int_S f(u,w)|\langle u,w\rangle|^{-\frac{n}{2}+i\sigma}|\langle v,w\rangle|^{-\frac{n}{2}-i\sigma}dw =: (T_\sigma f)(u,v),\ \forall\sigma\neq 0.$$

This integral does not converge absolutely, it must be considered as obtained by analytic continuation.

**Definition 1.6.** *A symbolic calculus on $X$ is a linear map $Op_\sigma : L^2(X) \to \mathcal{L}(L^2(\widetilde{S}))$ such that for every $f \in L^2(X)$ the function $(T_\sigma f)(u,v)$ is the kernel of the Hilbert-Schmidt operator $Op_\sigma(f)$ acting on $L^2(\widetilde{S})$.*

**Definition 1.7.** *The product $\#_\sigma$ on $L^2(X)$ is defined by*

$$Op_\sigma(f\sharp_\sigma g) = Op_\sigma(f)\circ Op_\sigma(g),\ \forall f,g\in L^2(X).$$

We thus have

- The product $\sharp_\sigma$ is associative.
- $\|f\sharp_\sigma g\|_2 \leq \|f\|_2\cdot\|g\|_2$, for all $f,g\in L^2(X)$.
- $Op_\sigma(L_x f) = \pi^+_{-\frac{n}{2}+i\sigma}(x)\,Op_\sigma(f)\,\pi^+_{-\frac{n}{2}+i\sigma}(x^{-1})$, so $L_x(f\sharp_\sigma g) = (L_x f)\sharp_\sigma(L_x g)$, for all $x\in G$, where $L_x$ denotes the left translation by $x\in G$ on $L^2(X)$.

This non-commutative product can be described explicitly:

$$(f\sharp_\sigma g)(u,v) = d(\sigma)\int_S\int_S f(u,x)g(y,v)|[u,y,x,v]|^{-\frac{n}{2}+i\sigma}d\mu(x,y), \tag{2}$$

where $d\mu(x,y) = |\langle x,y\rangle|^{-n}dxdy$ is a $G$-invariant measure on $\widetilde{S}\times\widetilde{S}$ for the $G$-action (1), and $[u,y,x,v] = \dfrac{\langle u,x\rangle\langle y,v\rangle}{\langle u,v\rangle\langle x,y\rangle}$.

On the space $L^2(X)$ there exists an (family of) involution $f \to f^*$ given by : $Op_\sigma(f^*) =: Op_\sigma(f)^*$. Notice that the correspondance $f \to Op_\sigma(\bar{f}^*)$ is what one calls in pseudo-differential analysis "anti-standard symbolic calculus". The link between symbols of standard and anti-standard calculus in the setting of the para-Hermitian symmetric space $X$ has been made explicit in [7] Corollary 1.4, see also Section 3.

Obviously we have $(f \natural_\sigma g)^* = g^* \natural_\sigma f^*$ and with the above product and involution, the Hilbert space $L^2(X)$ becomes a Hilbert algebra.

## 2. The structure of the subalgebra of $K$-invariant functions in $L^2(X)$

Let $\mathcal{A}$ be the subspace of all $K$-invariant functions in $L^2(X)$.

**Theorem 2.1.** *The subset $\mathcal{A}$ is a closed subalgebra of $L^2(X)$ with respect to the product $\natural_\sigma$.*

This statement clearly follows from the covariance of the symbolic calculus $Op_\sigma$, namely: $L_x(f \natural_\sigma g) = (L_x f) \natural_\sigma (L_x g)$, for all $x \in G, f, g \in L^2(X)$.

**Theorem 2.2.** *Let $n > 2$, then the subalgebra $\mathcal{A}$ is commutative.*

*Proof.* For a function $f \in L^2(X)$ we set $\check{f}(u,v) = f(v,u)$. The map $f \to \check{f}$ is a linear involution. Indeed,

$$(f \natural_\sigma g)(u,v) = d(\sigma) \int_S \int_S \check{f}(x,u) \check{g}(v,y) |[u,y,x,v]|^{-\frac{n}{2}+i\sigma} d\mu(x,y).$$

Permuting $x$ and $y$ and $u$ and $v$ respectively, we get

$$(f \natural_\sigma g)(v,u) = d(\sigma) \int_S \int_S \check{g}(u,x) \check{f}(y,v) |[v,x,y,u]|^{-\frac{n}{2}+i\sigma} d\mu(x,y).$$

But $|[v,x,y,u]| = |[u,y,x,v]|$, therefore $(f \natural_\sigma g)\check{} = \check{g} \natural_\sigma \check{f}$.

On the other hand, given a couple $(u,v) \in \widetilde{S} \times \widetilde{S}$ there exists an element $k \in K$ such that $k.(u,v) = (v,u)$. Geometrically $k$ can be seen as a rotation of angle $\pi[2\pi]$ around the axis defined by the bisectrix of vectors $u$ and $v$ in the plane they generate. Of course, such a $k$ exists for an arbitrary couple $(u,v)$ only if $n > 2$.

Hence for every $f \in \mathcal{A}$ we have $f = \check{f}$ and therefore $f \natural_\sigma g = g \natural_\sigma f$, for $f, g \in \mathcal{A}$. $\square$

## 3. Irreducible self-adjoint idempotents of $\mathcal{A}$

We begin with a **reduction theorem** for the multiplication and involution in $L^2(X)$.

As usual, we shall identify $L^2(X)$ with $L^2(\widetilde{S} \times \widetilde{S}; |\langle x,y \rangle|^{-n} dxdy)$. If $\phi \in L^2(X)$ we shall write $\phi(u,v) = |\langle u,v \rangle|^{n/2-i\sigma} \phi_o(u,v)$. Then $\phi_o \in L^2(\widetilde{S} \times \widetilde{S}; dsdt) = L^2(\widetilde{S} \times \widetilde{S})$, and therefore the map $\phi \to \phi_o$ is an isomorphism.

**Theorem 3.1.** *Under the isomorphism $\phi \to \phi_o$ the product $\#_\sigma$ translates into*

$$\phi_o \#_\sigma \psi_o (u,v) = d(\sigma) \int_S \int_S \phi_o(u,x) \, \psi_o(y,v) \, |\langle x,y \rangle|^{-n/2-i\sigma} dxdy$$

*and the involution becomes:*

$$\phi_0^*(u,v) = \overline{d(\sigma)}^2 \int_S \int_S \bar{\phi}_0(x,y) \left( |\langle x,v \rangle| |\langle u,y \rangle| \right)^{-n/2+i\sigma} dxdy.$$

The proof is straightforward. So we have translated the algebra structure of $L^2(X)$ to $L^2(\widetilde{S} \times \widetilde{S})$.

Let $\phi$ be an irreducible self-adjoint idempotent in $\mathcal{A}$. We shall give an explicit formula for the $\phi_o$-component of $\phi$.

Consider the decomposition of the space $L^2(\widetilde{S}) = \oplus_{\ell \in 2\mathbb{N}} V_\ell$, where $V_\ell$ is the space of harmonic polynomials on $\mathbb{R}^n$, homogeneous of even degree $\ell$.

Then the space $L^2(\widetilde{S} \times \widetilde{S})$ decomposes into a direct sum of tensor products $\oplus_{\ell, m \in 2\mathbb{N}} V_\ell \otimes \bar{V}_m$ and consequently $L^2_K(\widetilde{S} \times \widetilde{S}) = \oplus_{\ell \in 2\mathbb{N}} (V_\ell \otimes \bar{V}_\ell)^K$, where the sub(super-)script $K$ means: "the $K$-invariants in".

Let $\dim V_\ell = d$ and $f_1, \ldots, f_d$ be an orthonormal basis of $V_\ell$. Then the function $\theta_\ell(u, v) = \sum_{i=1}^d f_i(u) \bar{f}_i(v)$, that is the reproducing kernel of $V_\ell$, is, up to a constant, the $K$-invariant element of $V_\ell \otimes \bar{V}_\ell$.

**Theorem 3.2.** *Let $\phi(u, v) = |\langle u, v \rangle|^{n/2 - i\sigma} \phi_o(u, v)$ be an irreducible self-adjoint idempotent in $\mathcal{A}$. Then there exist complex numbers $c(\sigma, \ell)$ such that for any $\ell \in 2\mathbb{N}$ one has*

$$\phi_o(u, v) = c(\sigma, \ell)\, \theta_\ell(u, v).$$

*For different $\ell$ and $\ell'$ the idempotents annihilate each other. Moreover they span $\mathcal{A}$.*

*Proof.* Firstly we shall show that $\theta_\ell$ satisfies the condition

$$\theta_\ell \#_\sigma \theta_\ell = a(\sigma, \ell)\, \theta_\ell$$

for some constant $a(\sigma, \ell)$. Indeed,

$$\begin{aligned} & d(\sigma) \int_S \int_S \theta_\ell(u, x)\, \theta_\ell(y, v)\, |\langle x, y \rangle|^{-\frac{n}{2} - i\sigma} dx dy \\ = \quad & d(\sigma)\, e_\ell(\sigma) \int_S \theta_\ell(u, y)\, \theta_\ell(y, v)\, dy = d(\sigma)\, e_\ell(\sigma)\, \theta_\ell(u, v) \end{aligned}$$

by the intertwining relation (apply $A_{-\frac{n}{2} + i\sigma}$ to $\theta_\ell(., x)$):

$$\int_S \theta_\ell(u, x)\, |\langle x, y \rangle|^{-\frac{n}{2} - i\sigma} dx = e_\ell(\sigma)\, \theta_\ell(u, y)$$

where $e_\ell(\sigma) = \int_S \dfrac{\theta_\ell(e_1, x)}{\theta_\ell(e_1, e_1)} |x_1|^{-\frac{n}{2} - i\sigma}\, dx$.

Observe that $\dfrac{\theta_\ell(e_1, x)}{\theta_\ell(e_1, e_1)}$ is a spherical function on $\widetilde{S}$ with respect to $M$ of the form $a_\ell\, C_\ell^{\frac{n-2}{2}}(|x_1|)$ where $C_\ell^{\frac{n-2}{2}}(u)$ is a Gegenbauer polynomial and

$$a_\ell^{-1} = C_\ell^{\frac{n-2}{2}}(1) = 2^\ell \frac{\Gamma(\frac{n-2}{2} + \ell)}{\Gamma(\frac{n-2}{2}) \ell!}.$$

See for instance [9], Chapter IX, §3. Notice that $\theta_\ell(e_1, e_1) = \dim V_\ell = \dfrac{(n + \ell - 1)!}{(n-1)! \ell!} \neq 0$. The integral defining $e_\ell(\sigma)$ does not converge absolutely, but has to be considered as the meromorphic extension of an analytic function. Poles only occur in half-integer points on the real axis. So we have to restrict (and we do) to $\sigma \neq 0$.

So we have $\theta_\ell \#_\sigma \theta_\ell = d(\sigma)\, e_\ell(\sigma)\, \theta_\ell$ and hence $\varphi_\ell = [d(\sigma)\, e_\ell(\sigma)]^{-1}\, \theta_\ell$ is the $\phi_o$-component of an idempotent in $\mathcal{A}$. Furthermore $\theta_\ell \#_\sigma \theta_{\ell'} = 0$ if $\ell \neq \ell'$. Clearly $\varphi_\ell$ is self-adjoint, since $|d(\sigma)|^{-2} = |e_\ell(\sigma)|^2$, being equal to the constant $c(\sigma)$ from Section 1.

So the $\varphi_\ell$ are mutually orthogonal idempotents in the algebra $L^2_K((\widetilde{S}\times\widetilde{S}); dsdt)$ and span this space. The theorem now follows easily. $\square$

**Remark** The constant $e_\ell(\sigma)$ can of course be computed. Apply e.g. [3], Section 7.31, we get, by meromorphic continuation:

$$\begin{aligned} e_\ell(\sigma) &= a_\ell \int_S C_\ell^{\frac{n-2}{2}}(|x_1|)\,|x_1|^{-\frac{n}{2}-i\sigma}\,dx \\ &= 2\,a_\ell \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\sqrt{\pi}} \int_0^1 u^{-\frac{n}{2}-i\sigma}\,(1-u^2)^{\frac{n-2}{2}}\,C_\ell^{\frac{n-2}{2}}(u)\,du \\ &= 2^{-2\ell}\,\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}}\cdot\frac{\Gamma(n-2+\ell)}{\Gamma(n-2)}\cdot\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-2}{2}+\ell)}\cdot\frac{\Gamma(-\frac{n}{2}-i\sigma+1)\Gamma(\frac{-\frac{n}{2}-i\sigma-\ell+1}{2})}{\Gamma(-\frac{n}{2}-i\sigma-\ell+1)\Gamma(\frac{\frac{n}{2}-i\sigma+\ell}{2})}. \end{aligned}$$

## 4. The strucure of the Hilbert algebra $L^2(X)$

We now turn to the full algebra $L^2(X)$. We again reduce the computations to $L^2(\widetilde{S}\times\widetilde{S})$. In a similar way as for $\mathcal{A}$ we get:

**Lemma 4.1.** *If $\phi_o \in V_\ell \otimes \overline{V}_m$, $\psi_o \in V_{\ell'}\otimes\overline{V}_{m'}$ then*

$$\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } m \neq \ell' \\ \text{in} & V_\ell\otimes\overline{V}_{m'} \text{ if } m=\ell'. \end{cases}$$

*More precisely we have the following result. Let $(f_i)$, $(g_j)$, $(k_l)$ be orthonormal bases of $V_\ell$, $V_m$ and $V_{m'}$ respectively, and $\phi_o(u,v) = f_i(u)\overline{g}_j(v)$, $\psi_o(u,v) = g_{j'}(u)\overline{k}_l(v)$, then*

$$\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } j\neq j' \\ d(\sigma)\,e_m(\sigma)\,f_i(u)\overline{k}_l(v) & \text{if } j=j'. \end{cases}$$

The proof is again straightforward and uses the intertwining relation:

$$\int_S |\langle x,y\rangle|^{-n/2-i\sigma}\,g_{j'}(y)dy = e_m(\sigma)\,g_{j'}(x).$$

**Theorem 4.2.** *The irreducible self-adjoint idempotents of $L^2(\widetilde{S}\times\widetilde{S})$ are given by*

$$e_f^\ell(u,v) = \{d(\sigma)\,e_\ell(\sigma)\}^{-1}\cdot f(u)\,\overline{f}(v)$$

*with $f\in V_\ell$, $\|f\|_{L^2(\widetilde{S})}=1$ and $\ell$ even. The left ideal generated by $e_f^\ell$ is equal to $L^2(\widetilde{S})\otimes\overline{f}$.*

The proof is by application of Lemma (4.1)

**Remarks**

(1) The minimal right ideals are obtained in a similar way.
(2) The minimal two-sided ideal generated by $L^2(\widetilde{S}\times\widetilde{S})\cdot e_f^\ell$ is the full algebra $L^2(\widetilde{S}\times\widetilde{S})$.
(3) The closure of $\bigoplus_{\ell\in 2\mathbb{N}} V_\ell\otimes\overline{V}_\ell$ is a $H^*$-subalgebra of $L^2(\widetilde{S}\times\widetilde{S})$. The minimal left ideals are here $V_\ell\otimes\overline{f}$ ($f\in V_\ell$, $\|f\|_{L^2(\widetilde{S})}=1$); they are generated by the $e_f^\ell$ as above. The minimal two-sided ideal generated by $V_\ell\otimes\overline{f}$ is equal to $V_\ell\otimes\overline{V}_\ell$.

## 5. The case of a general para-hermitian space

It is not necessary to assume rank $X = 1$ in order to show that $\mathcal{A}$ is commutative. Theorem 3.2 is also valid mutatis mutandis in the general case since $(K, K\cap H)$ is a Gelfand pair, and it clearly implies the commutativity of $\mathcal{A}$. To the general construction of the product and the involution we shall return in another paper.

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